Semistable Comparison

- $K / K_0 = W(k)[\frac{1}{p}]$ fin. tot. ram. $G_K = \mathrm{Gal}(\bar{K}/K)$

X: proper semist / $\mathcal{O}_K$.

Assume trivial divisor @ ∞: $X_{tr} = X_K$.

(o/w, e.g. $A = \mathcal{O}_K[x^{\pm 1}, y_1, y_2, z_1, z_2]/(y_1 y_2 - \varpi)$

$(\mathrm{Spec} A)_{tr} = \mathrm{Spec} A_K \setminus D_K$, $D = \{z_1 z_2 = 0\}$).

Thm $\exists$ nat'l G_K-equiv. isom.

$$\begin{array}{ccc} H^i_{et}(X_{\bar{K}}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} B_{st} & \simeq & H^i_{HK}(X) \otimes_{K_0} B_{st} \quad \circlearrowleft \varphi, N \\ \downarrow & & \downarrow \\ H^i_{et}(\quad '' \quad) \otimes B_{dR} & \simeq & H^i_{dR}(X_K) \otimes_K B_{dR} \quad \mathrm{Fil}. \end{array}$$

Rmk When non-triv. div. @ ∞, $X_{tr, \bar{K}}$ in place of $X_{\bar{K}}$.

Idea Study LES

$$\cdots \to (H^{i-1}_{dR}(X_K) \otimes B^+_{dR})/F^r \to H^i_{syn}(X_{\mathcal{O}_{\bar{K}}}, r)_{\mathbb{Q}} \qquad (*)$$

$$\to (H^i_{HK}(X) \otimes_{K_0} B^+_{st})^{\varphi = p^r, N=0} \longrightarrow (H^i_{dR}(X_K) \otimes_K B^+_{dR})/F^r \to \cdots$$

via Fin. dim. BC sp's. + EM map.

§1. LES $(*)$

Not'n • For V DVR, V: triv log. str. $V^\times$: can. log.

V^0: log by $\mathbb{N} \to V$, $1 \mapsto 0$.

$(X_n / W_n(k))_{cr}$: log-cry. site.

obj's : $(\underset{\text{ét}/X_n}{U}, \underset{\log}{M}) \xrightarrow{f} (T, M_T)$ exact clsd imm. + PD-str.

$f^* M_T \simeq M$

J_{X_n/W_n} : PD-ideal $\quad u : (X_n/W_n)_{cr} \to X_{ét}$.

- $R\Gamma_{cr}(X, J^{[r]})_n := R\Gamma(X_{ét}, Ru_* J^{[r]}_{X_n/W_n})$ → rth div. pow.

$R\Gamma_{cr}(\quad " \quad) := \mathrm{holim}_n (\quad " \quad)_n$.

- $R\Gamma_{syn}(X, r)_n := [R\Gamma_{cr}(X, J^{[r]})_n \xrightarrow{p^r - \varphi} R\Gamma_{cr}(X_n)]$

$\simeq [R\Gamma_{cr}(X)_n \xrightarrow{(p^r - \varphi, \mathrm{can})} R\Gamma_{cr}(X)_n \oplus R\Gamma_{cr}(X, \mathcal{O}/J^{[r]})_n]$

- $R\Gamma_{cr}(X_{\mathcal{O}_F}, J^{[r]})$ & $R\Gamma_{syn}(X_{\mathcal{O}_F}, r)$ similarly.

$\boxed{\text{Lem}}$ $R\Gamma_{syn}(X_{\mathcal{O}_F}, r)_{\mathbb{Q}}$

"$\simeq$" $[[R\Gamma_{HK}(X) \otimes_{F_0} B^+_{st}]^{\varphi = p^r, N=0} \xrightarrow{\iota_{dR}} (R\Gamma_{dR}(X_K) \otimes_K B^+_{dR}) / F^r]$

- $R\Gamma_{HK}(X) := R\Gamma_{cr}(X_k / W(k)^0)_{\mathbb{Q}}$.

- ι_{dR} : $R\Gamma_{HK}(X) \to R\Gamma_{dR}(X_K)$ Hyodo-Kato map.

$R\Gamma_{dR}(X_K)/F^m \simeq R\Gamma_{cr}(X, \mathcal{O}_{X/\mathcal{O}_K^\times} / J^{[m]}_{X/\mathcal{O}_K^\times})_{\mathbb{Q}}$

$\overset{\gamma_m ??}{\xleftarrow{\sim}} R\Gamma_{cr}(X, \mathcal{O}_{X/W(k)} / J^{[m]}_{X/W(k)})_{\mathbb{Q}}$

$r^{PD}_{\varpi} = \{ \sum_{i=0}^{\infty} a_i \frac{X_0^i}{\lfloor i/e \rfloor !} : a_i \in W(k),\ a_i \to 0 \}$

$N(X_0) = -X_0$, log. str. gen. by X_0.

$j_0 : r^{PD}_{\varpi} \twoheadrightarrow W(k)^{0}$, $x_0 \mapsto 0$, $j_{\varpi} : r^{PD}_{\varpi} \twoheadrightarrow \mathcal{O}_K^{\times}$, $x_0 \mapsto \varpi$

$$j_0^* : R\Gamma_{cr}(X/r^{PD}_{\varpi})_{\mathbb{Q}} \to R\Gamma_{HK}(X)$$

$\exists !$ section s : $R\Gamma_{HK}(X) \to R\Gamma_{cr}(X/r^{PD}_{\varpi})_{\mathbb{Q}}$

Dwork's trick

$$r^{PD}_{\varpi} \otimes_{W(k)} R\Gamma_{HK}(X) \xrightarrow[s]{\simeq} R\Gamma_{cr}(X/r^{PD}_{\varpi})_{\mathbb{Q}}.$$

$\Rightarrow$ Hyodo – Kato map

$$\iota_{dR} : R\Gamma_{HK}(X) \xrightarrow{s} R\Gamma_{cr}(X/r^{PD}_{\varpi})_{\mathbb{Q}} \xrightarrow{j_{\varpi}^*} R\Gamma_{cr}(X/\mathcal{O}_K^{\times})_{\mathbb{Q}} \simeq R\Gamma_{dR}(X_K).$$

• $R\Gamma_{syn}(X,r)_{\mathbb{Q}} \simeq \left[R\Gamma_{cr}(X)_{\mathbb{Q}} \xrightarrow{(1-\frac{\varphi}{p^r},\, \gamma_r)} R\Gamma_{cr}(X)_{\mathbb{Q}} \oplus R\Gamma_{dR}(X_K)/F^r \right]$

$$\underset{①}{\xrightarrow{\sim}} \left[\begin{array}{ccc} R\Gamma_{cr}(X/r^{PD}_{\varpi})_{\mathbb{Q}} & \xrightarrow{(1-\frac{\varphi}{p^r},\, \cdot)} & R\Gamma_{cr}(X/r^{PD}_{\varpi})_{\mathbb{Q}} \oplus R\Gamma_{dR}(X_K)/F^r \\ \downarrow N & & \downarrow (N,0) \\ R\Gamma_{cr}(X/r^{PD}_{\varpi})_{\mathbb{Q}} & \xrightarrow{1-\frac{\varphi}{p^{r-1}}} & R\Gamma_{cr}(X/r^{PD}_{\varpi})_{\mathbb{Q}} \end{array} \right]$$

$$\underset{②}{\xleftarrow[s]{\sim}} \left[\begin{array}{ccc} R\Gamma_{HK}(X) & \xrightarrow{(1-\frac{\varphi}{p^r},\, \iota_{dR})} & R\Gamma_{HK}(X) \oplus R\Gamma_{dR}(X_K)/F^r \\ \downarrow N & & \downarrow N \\ R\Gamma_{HK}(X) & \xrightarrow{1-\frac{\varphi}{p^{r-1}}} & R\Gamma_{HK}(X) \end{array} \right]$$

① : dist. Δ $\quad R\Gamma_{cr}(X) \to R\Gamma_{cr}(X/r^{PD}_{\varpi}) \xrightarrow{N} R\Gamma_{cr}(X/r^{PD}_{\varpi})$

② :

$$\begin{array}{ccc} H^i_{HK}(X) & \xrightarrow{p^r-\varphi} & H^i_{HK}(X) \\ \uparrow & & \uparrow \\ r^{PD}_{\varpi} \otimes_W H^i_{HK}(X) & \xrightarrow{p^r-\varphi} & r^{PD}_{\varpi} \otimes_W H^i_{HK}(X) \end{array}$$

Vert. maps induce isom. on $\ker(p^r-\varphi)$ & $\mathrm{coker}(p^r-\varphi)$.

$$\Rightarrow R\Gamma_{syn}(X,r)_{\mathbb{Q}} \simeq \left[\, [R\Gamma_{HK}(X)]^{\varphi=p^r, N=0} \xrightarrow{\iota_{dR}} R\Gamma_{dR}(X_K)/F^r \right]$$

Beilinson : $R\Gamma_{HK}$ via cx of (φ,N)-mod's.

Nekovar-Niziol : geometrizing above. $(K\to\bar{K}) \Rightarrow$ Lem.

$\boxed{\text{Lem}}$ $R\Gamma_{syn}(X_{\mathcal{O}_{\bar{K}}}, r)_{\mathbb{Q}}$

$$\text{“}\simeq\text{”} \left[\, [R\Gamma_{HK}(X)\otimes_{K_0} B^+_{st}]^{\varphi=p^r, N=0} \xrightarrow{\iota_{dR}} (R\Gamma_{dR}(X_K)\otimes_K B^+_{dR})/F^r \right]$$

Fact • $H^i((R\Gamma_{dR}(X_K)\otimes_K B^+_{dR})/F^r) \simeq H^i_{dR}(X_K)\otimes_K B^+_{dR}/F^r$.

(by degen. of Hodge-de Rham spectral seq.)

• $H^i[R\Gamma_{HK}(X)\otimes_{K_0} B^+_{st}]^{\varphi=p^r, N=0} \simeq (H^i_{HK}(X)\otimes_{K_0} B^+_{st})^{\varphi=p^r, N=0}$:

For (φ,N)-mod M / K_0.

$$0\to (M\otimes_{K_0} B^+_{cr})^{\varphi=p^r} \longrightarrow M\otimes_{K_0} B^+_{cr} \xrightarrow{1-\frac{\varphi}{p^r}} M\otimes_{K_0} B^+_{cr} \to 0$$

exact

& if $N(M)=0$.

$$0\to M\otimes_{K_0} B^+_{cr} \to M\otimes_{K_0} B^+_{st} \xrightarrow{N} M\otimes_{K_0} B^+_{st} \to 0. \quad \text{exact}$$

$\Rightarrow$ LES ⊛

§2. Fin. dim. BC sp's.

Fin. dim. $\mathbb{W}$: $\mathrm{Dim}\,\mathbb{W} = (\mathrm{dim}\,\mathbb{W}, \mathrm{ht}\,\mathbb{W})$

$= (d, \dim_{\mathbb{Q}_p} V_1 - \dim_{\mathbb{Q}_p} V_2)$

$$\begin{tikzcd}
& 0 \arrow[r] & V_2 \arrow[dr] & & & \\
0 \arrow[r] & V_1 \arrow[r] & & Y \arrow[r] \arrow[dr] & \mathbb{W}^d \arrow[r] & 0 \\
& & & & \mathbb{W} \arrow[r] & 0
\end{tikzcd}$$

- $f : \mathbb{W}_1 \to \mathbb{W}_2$

$\Rightarrow \operatorname{Dim} \mathbb{W}_1 = \operatorname{Dim} \ker f + \operatorname{Dim} \operatorname{Im} f$, $\quad \operatorname{Dim} \mathbb{W}_2 = \operatorname{Dim} \operatorname{Im} f + \operatorname{Dim} \operatorname{coker} f$.

<u>Fact</u> (1) $\dim \mathbb{W} = 0 \Rightarrow \operatorname{ht} \mathbb{W} \geq 0$.

(2) $f : \mathbb{W}_1 \to \mathbb{W}_2$ $\quad \operatorname{Ker}(f)(C)$ & $\operatorname{coker}(f)(C)$ fin. $/\mathbb{Q}_p$.

$\Rightarrow \dim \mathbb{W}_1 = \dim \mathbb{W}_2$

(3) $\mathbb{W}_2$: succ. ext'n of $\mathbb{V}$'s $\quad \mathbb{W}_1 \hookrightarrow \mathbb{W}_2$

$\Rightarrow \operatorname{ht} \mathbb{W}_1 \geq 0$.

(4) $\mathbb{W}_2$: succ. ext'n of $\mathbb{V}$'s $\quad f : \mathbb{W}_1 \to \mathbb{W}_2$.

$\ker(f)(C)$ fin. $/\mathbb{Q}_p$ $\quad \dim \mathbb{W}_1 = \dim \mathbb{W}_2 \Rightarrow f$ surj.

- $D = (D_{st}, D_{dR}, \lambda)$ $\quad D_{st}$: (φ, N)-mod $/ k_0$.

D_{dR} : fil. mod $/ K$ $\quad \lambda : D_{st} \to D_{dR}$ $\quad k_0$-lin.

$\mathbb{X}^r_{st}(D) = (t^{-r} B^+_{st} \otimes_{k_0} D_{st})^{\varphi=1, N=0}$ $\quad \mathbb{X}^r_{dR}(D) = (t^{-r} B^+_{dR} \otimes_K D_{dR}) / F^0$

- If $F^{r+1} D_{dR} = 0$ & $r \geq$ slopes of φ $\;(r(D) :=$ min. such $r)$.

$\operatorname{Dim} \mathbb{X}^r_{st}(D) = (r \dim_{k_0} D_{st} - t_N(D_{st}), \; \dim_{k_0} D_{st})$

$\operatorname{Dim} \mathbb{X}^r_{dR}(D) = (r \dim_K D_{dR} - t_H(D_{dR}), \; 0)$

- If $r \geq r(D)$ & $K \otimes_{k_0} D_{st} \xrightarrow[\lambda]{\sim} D_{dR}$,

$\mathbb{W}_{st}(D) := \ker(\mathbb{X}^r_{st}(D) \xrightarrow{\lambda} \mathbb{X}^r_{dR}(D))$ $\quad$ indep of r.

- Not'n : $\mathbb{W}(\mathbb{C}) = W$.

$\boxed{\text{Lem}}$ Sps $\lambda : K \otimes_{K_0} D_{st} \to D_{dR}$ isom. & $t_H(D_{dR}) = t_N(D_{st})$.

(1) TFAE : (a) $V_{st}(D)$ fin. / $\mathbb{Q}_p$.

(b) $X_{st}^r(D) \to X_{dR}^r(D)$ surj. for $r = r(D)$ ((b') $\forall r \geq r(D)$).

Cond's in (1) imply :

(2) $\dim_{\mathbb{Q}_p} V_{st}(D) = rk(D)$.

(3) D is wkly adms.

pf) (1) Follows from Fact (4).

(3) If $\exists D' \subset D$ s.t. $t_N(D') < t_H(D')$. Then for $r \gg 0$

$\dim \mathbb{W}_{st}(D') \geq 1$. $\Rightarrow$ ~~(a)~~ (a) $\blacksquare$

$\boxed{\text{Prop 1}}$ Given $D^i = (D^i_{st}, D^i_{dR}, \lambda^i)$ for each i s.t. $F^{i+1} D^i_{dR} = 0$.

Sps for each r, have LES

$$\cdots \to H^i(r) \to X^r_{st}(D^i) \to X^r_{dR}(D^i) \to H^{i+1}(r) \to \cdots$$

where $H^i(r)$ fin. / $\mathbb{Q}_p$ if $i \leq r$, Then :

(1) $\lambda^i : K \otimes_{K_0} D^i_{st} \xrightarrow{\sim} D^i_{dR}$ $\forall i$.

(2) D^i is wkly adms. $\forall i$.

(3) $0 \to H^i(r) \to X^r_{st}(D^i) \to X^i_{dR}(D^i) \to 0$ exact if $\underline{i \leq r}$.

pf) $\forall r \gg 0$, $\dim \mathbb{X}^r_{st}(D^i) = \dim \mathbb{X}^r_{dR}(D^i)$ by Fact (2).

$\Rightarrow \dim_{K_0} D^i_{st} = \dim_K D^i_{dR}$, $t_H(D^i) = t_N(D^i)$.

(1) $D' := \operatorname{Im}(\lambda^i) \subset D^i_{dR}$, $\operatorname{coker}(\mathbb{X}^r_{st}(D^i) \to X^i_{dR}(D^i)) \twoheadrightarrow \mathbb{X}^r_{dR}(D^i/D')$

If $r \geq i+1$, $\mathrm{coker}(C) \hookrightarrow H^{i+1}(r)$ fin. $/ \mathbb{Q}_p$.

$\Rightarrow D^i = D'$.

(2) By (1) & Lem.

$r(D^i) \leq i$. For $i \leq r$, Lem $\Rightarrow$

$0 \to H^i(r) \to X_{st}^r(D^i) \to X_{dR}^r(D^i) \to 0$ exact. $\Rightarrow$ (3). $\blacksquare$

§3. FM map

- A : étale $/ \mathcal{O}_K[X_1^{\pm 1}, \dots, X_a^{\pm 1}, X_{a+1}, \dots, X_{a+b}] / (X_{a+1} \cdots X_{a+b} - \varpi^h)$

$R = \hat{A}$ $\quad G_R = \mathrm{Gal}(\bar{R}[\frac{1}{p}] / R[\frac{1}{p}])$

$E^{PD}_{\bar{R},n}$: log-PD envel. of $\bar{R}_n$ in $A^+_{cr}(\bar{R})_n \otimes R^+_{\varpi}$

$$\Omega_{E^{PD}_{\bar{R},n}} := E^{PD}_{\bar{R},n} \otimes_{R^+_{\varpi,n}} \Omega_{R^+_{\varpi,n}}$$

$$\alpha^{FM}_{r,n} : \mathrm{Syn}(R,r)_n = [\, F^r \Omega^\bullet_{R^{PD}_{\varpi,n}} \xrightarrow{p^r - p^\bullet \varphi} \Omega^\bullet_{R^{PD}_{\varpi,n}} \,]$$

$$\longrightarrow C(G_R, [\, F^r \Omega^\bullet_{E^{PD}_{\bar{R},n}} \xrightarrow{p^r - p^\bullet \varphi} \Omega^\bullet_{E^{PD}_{\bar{R},n}} \,])$$

$$\xleftarrow{\sim} C(G_R, [\, F^r A_{cr}(\bar{R})_n \xrightarrow{p^r - \varphi} A_{cr}(\bar{R})_n \,])$$

$$\xleftarrow{\sim} C(G_R, \mathbb{Z}/p^n(r)')$$

$F^r A_{cr}(\bar{R})_n \simeq F^r \Omega^\bullet_{E^{PD}_{\bar{R},n}}$

$0 \to \mathbb{Z}_p(r)' \to F^r A_{cr}(\bar{R}) \xrightarrow{p^r - \varphi} A_{cr}(\bar{R}) \to 0$ exact

$\frac{1}{p^{a(r)}} \mathbb{Z}_p(r)$, $\quad a(r) = \lfloor \frac{r}{p-1} \rfloor$

$\boxed{\text{Lem}}$ If k has enough roots of unity, $\alpha^{FM}_{r,n}$ is p^c-equal to $\alpha^{Laz}_{r,n}$ for const. c indep. of k.

Idea diagram chase: $\mathrm{Kos}(\varphi, d) \xrightarrow{t^\bullet} \mathrm{Kos}(\varphi, \mathrm{Lie}\,T_R) \underset{\sim}{\xleftarrow{Laz}} \mathrm{Kos}(\varphi, T_R)$

$$\xrightarrow{\sim} C(T_R) \xrightarrow[\text{Inf'n}]{} C(G_R).$$

- $\mathcal{S}_n(r)$: ét sheaf'n of $(U \xrightarrow{\text{ét}} X) \mapsto R\Gamma_{syn}(U, r)_n$

$\iota : X_k \hookrightarrow X$, $j : X_K \hookrightarrow X$.

$\alpha^{FM}_{r,n} : \mathcal{S}_n(r)_X \to \iota^* Rj_* \mathbb{Z}/p^n(r)'_{X_K}$.

$\boxed{\text{Cor}}$ X : base change of semist schm $/ \mathcal{O}_{K_1}$ for some $K_1 \subset K$. Sps k has enough roots of unity. $i \le r$. Then $\alpha^{FM}_{r,n}$ is p^c-q.isom. for some c indep. of k.

pf) A : loc. chart of X as before.
A^h : p-adic henzl'n of A.

$$\begin{array}{ccccc} H^i(\mathrm{Syn}(A, r)_n) & \xrightarrow{\alpha^{FM}} & H^i(G_{A^h}, \mathbb{Z}/p^n(r)') & \to & H^i(A^h[\tfrac{1}{p}]_{\text{ét}}, \mathbb{Z}/p^n(r)') \\ \| & & S\downarrow & & S\downarrow\ \ast \\ H^i(\mathrm{Syn}(\hat{A}, r)_n) & \xrightarrow[\triangle]{\alpha^{FM}} & H^i(G_{\hat{A}}, \mathbb{Z}/p^n(r)') & \underset{\triangle}{\to} & H^i(\hat{A}[\tfrac{1}{p}]_{\text{ét}}, \mathbb{Z}/p^n(r)') \end{array}$$

Lem Elkik's alg'n thm. $K(\pi,1)$-lem. (Scholze)

- g is q-isom. by rigid GAGA :

$$\begin{array}{ccc} \mathrm{F\acute{e}t}(A^h) & \to & \mathrm{F\acute{e}t}(A^h[\tfrac{1}{p}]) \\ S1\downarrow & & g\downarrow \\ \mathrm{F\acute{e}t}(\hat{A}) & \to & \mathrm{F\acute{e}t}(\hat{A}[\tfrac{1}{p}]) \end{array}$$

fiber D in $D(\mathbb{Z})^{\geq 0}$.

affine anal. of proper base change (Gabber).

$\square$

$\Rightarrow$ Prop 2 For $r \geq i$, $H^i_{syn}(X_{\mathcal{O}_{\bar{F}}}, r)_{\mathbb{Q}} \xrightarrow[q\text{-}FM]{\sim} H^i_{et}(X_{\bar{F}}, \mathbb{Q}_p(r))$.

Prop 1 ($\tilde{r} = r$) & Prop 2 $\Rightarrow$ Thm.